Stabilizing strategy for delayed recycling systems with unstable dynamics

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Abstract: This work deals with the stabilization of recycling systems with unstable dynamics at both, the direct path as well as the recycling trajectory, also two different time-delays are considered in the internal dynamic. The main contribution of this work focuses to obtain a stabilizing strategy to the mentioned class of system. Even when in literature there are stability results for stable or partial unstable recycling system, it should be highlighted that there is not stability results when unstable dynamic is considered in both paths of the recycling process. The stability conditions of the proposed stabilizing controllers are obtained from a frequency domain analysis.

Keywords: delayed system, linear control, time-delay.

1. INTRODUCTION

Recycling systems are commonly found in chemical industry, for instance, in a typical plant formed by reactor/separator process, where reactants are recycled back to the reactor (Luyben et al. (1999)). They reuse the energy and the partially processed matter increasing the efficiency of the overall process. In recycling systems a partial feedback of the process output to the input induces a positive feedback, which can give rise to some undesirable effects. Luyben (1999) studied the effects of recycle path on dynamics process and their implications to plant-wide control. Taiwo (1986), discussed the robust control for recycling plants and proposed the concept of recycle compensation to recuperate inherent process dynamics, i.e. dynamics without recycle. Scali and F. Ferrari (1999) analyzed the problem under same idea. Similar approaches were extended by Lakshminarayanan and Takada (2001), and Kwok et al. (2001). It is known that when recycle path and time-delays occur, exponential terms appear in the direct and the recycling paths. In a state space representation, recycling systems with time-delay correspond to systems with delays in the input and the state variables. In Del-Muro-Cuellar et al. (2005) an approximated model to represent recycling systems by using discrete-time approach is proposed. In turn, such approximated models can be used for stability analysis or control design; Samyudia et al. (2000), Scali and F.Ferrari (1999), Astrom et al. (1994), Maza-Casas et al. (1999). A system with time-delay and open-loop unstable poles is notably more difficult to control than a system with only open-loop stable poles. For instance, the classical Smith Predictor cannot be used in the case of delayed unstable processes. Introducing recycle in such system would lead to a more difficult (although interesting) problem. Existing modified schemes to Smith Predictor cannot directly be applied to this kind of systems. This is, because the open-loop recycling system is not a system with only one time-delay in the direct path but it is a system with internal delay term, i.e., an open loop infinite dimensional system. In order to counteract this problem, we consider here recycling systems composed of a plant with one unstable pole, several stable poles, a delay term in the direct path and a delayed unstable subsystem (with one unstable pole and several stable poles) in the recycling path.

Even when in literature there are stability results to stable recycling systems or partial unstable recycling system, it should be highlighted that there is not stability results when unstable dynamics are considered in both loops of the recycling process. In this work, a control scheme is proposed in order to stabilize and control delayed recycling previously depicted. The proposed scheme and controllers allows achieving the closed-loop stability, then, necessary and sufficient conditions are given for the existence of the proposed stabilizing scheme, which are obtained form a frequency domain approach. It is important to note that, the problem of the stabilization and control of delayed unstable plants even without recycle path is not completely solved. For instance, recent works Lee et al. (2010), Silva et al. (2005) deal with the stabilization and control of delayed systems with only one unstable pole.
The outline of the paper is as follows. In Section 2 the problem is formulated and the class of systems considered in this work is precised. The general idea of the solutions is also outlined in this section. The Section 3 presents the main results. Some simulation results are described in Section 4. Such results illustrate the performance of the proposed control strategy. Finally Section 5 presents some conclusions.

2. PROBLEM FORMULATION

Consider the class of recycling system shown in Fig. 1, which can be described as,
\[ Y(s) = [G_d(s) G_d(s)G_r(s)] \begin{bmatrix} U(s) \end{bmatrix}, \]
with,
\[ G_d(s) = G_1(s)e^{-\tau_1 s} = \frac{\alpha}{(s-a)(s+b_1)...(s+b_m)} e^{-\tau_1 s}, \]
\[ G_r(s) = G_2(s)e^{-\tau_2 s} = \frac{\beta}{(s-c)(s+d_1)...(s+d_n)} e^{-\tau_2 s}, \]
where \( G_d(s) \), and \( G_r(s) \) are transfer functions of the direct and the recycling paths, respectively; \( \tau_1, \tau_2 \geq 0 \) are the time-delays associated to \( G_d(s) \) and \( G_r(s) \). \( a, b_i, c, d_j \in \mathbb{R}^+ \) with \( i = 1, 2, ..., m \) and \( j = 1, 2, ..., n \), i.e., \( G_d(s) \) and \( G_r(s) \) are unstable. \( U(s) \) is the input process and \( Y(s) \) is the output process.

![Fig. 1. A process with recycle](image)

The transfer function of the recycling system (1) is given by,
\[ G_t(s) = \frac{Y(s)}{U(s)} = \frac{D_2\alpha e^{-\tau_1 s}}{D_1(s)D_2(s) - \alpha\beta e^{-(\tau_1+\tau_2)s}}, \]
where \( D_1(s) = (s-a)(s+b_1)...(s+b_m) \) and \( D_2(s) = (s-c)(s+d_1)...(s+d_n) \). Note that exponential terms appear explicitly in the numerator and the denominator of \( G_t(s) \). Stability of (3) is determined by the roots of its characteristic quasi-polynomial,
\[ Q(s) = D_1(s)D_2(s) - \alpha\beta e^{-(\tau_1+\tau_2)s}. \]
Note that the transcendent term in \( Q(s) \) induces an infinite number of roots. Then, for this kind of plants it is not an easy task to conclude about the dynamical behaviour (stability for instance) even in the uncontrolled plant case. Obviously, the related transfer function when the system is controlled with an output feedback becomes more complicated involving more than one transcendent term.

Let us to describe some ideas behind the proposed methodology. With reference to Fig. 1, if the signal \( \omega_1(s) \) were measured, then we could set,
\[ U(s) = (R(s) - \omega_1(s))C(s), \]
obtaining the system shown in Fig. 2, where \( R(s) \) is the input reference and \( C(s) \) is the controller. Then, it would be possible to design the controller, \( C(s) \), such that the closed-loop system is stable. In fact, the main aims of this research work can be summarized as follows i) Finding the stability conditions of the proposed feedback shown in Fig. 2. ii) Since \( \omega_1(s) \) is assumed as unmeasured internal system signal, an observer scheme to estimate the interest variable should be performed. In the present paper only the first goal is tackled i.e., the stability properties analysis derived from the proposed feedback shown in Fig. 2. The second part of the proposal consists in the design of the special estimation strategy for the signal \( \omega_1(s) \) which will be considered as future work.

![Fig. 2. Recycling delayed process with the proposed control law (5)](image)

3. MAIN RESULT

In order to implement the ideas developed in Fig. 2, let us assume that an adequate estimation of the signal \( \omega_1 \) is taken from an estimation strategy previously designed. In this way, in the following developments the stability properties of the closed-loop system shown in Fig. 2 are analyzed. The following result presents the stability properties for the mentioned closed-loop system when a Proportional-Derivative (PD) controller is considered using a frequency domain analysis. Later, a Proportional-Integral-Derivative (PID) control is also considered to stabilize the closed-loop system.

\[ \text{Theorem 1. Consider the feedback control shown in Fig. 2. Then, the delayed recycling system given by (1) can be stabilized with a PD controller given by,} \]
\[ C(s) = K_P + K_Ds, \]
\[ \text{or} \]
\[ C(s) = k_p(k_ds - 1) + 1, \]
with \( k_p, k_d > 0 \), if and only if,
\[ \theta < \frac{1}{a} + \frac{1}{c} + \sum_{i=1}^{n} \frac{1}{b_i} + \sum_{i=1}^{m} \frac{1}{d_i} - \sqrt{\frac{1}{a^2} + \frac{1}{b_i^2} + \sum_{i=1}^{n} \frac{1}{b_i^2} + \sum_{i=1}^{m} \frac{1}{d_i^2}}, \]
with \( \theta = \tau_1 + \tau_2 \)

\[ \text{Proof.} \]

The closed-loop transfer function of the system shown in Fig. 2 is given by,
\[ Y(s) = \frac{CG_d}{R(s) - (1 - C)G_dG_r}. \]
and the associated closed-loop characteristic equation is,
\[ 1 - (1 - C)G_dG_r = 0 \]  
(10)
or equivalently,
\[ 1 + \bar{C}G_dG_r = 0 \]  
(11)
with \( \bar{C} = C - 1 \). The characteristic equation (11) can be also obtained from the auxiliary closed-loop system consisting in an open-loop transfer function \( \bar{Q}(s) = \bar{C}G_dG_r \) with an unitary feedback. In this way, the stability conditions of the proposed auxiliary closed-loop system are equivalent to the required stability conditions for the characteristic equation (11).

The Nyquist stability criterion states that for a given open-loop system \( \bar{Q}(s) \), there is a stabilizing proportional control \( k \) such that the closed-loop system is stable iff \( 0 = N + P \), where \( P \) is the number of unstable roots in \( \bar{Q}(s) \) and \( N \) is the number of clockwise round trips to the point \((-1,0j)\) in the open-loop \( \bar{Q}(s) \) Nyquist diagram. It can be seen that for our particular case \( j\omega\) is \( \bar{C}j\omega G_d(j\omega)G_r(j\omega) \). From (11) it should satisfy \( N = -2 \), i.e., in order to achieve closed-loop stability the Nyquist plot should encircle twice the critical point \((-1,0j)\) in counter-clockwise direction. Therefore, the selection of the derivative should be proposed to obtain a Nyquist trajectory such that the closed loop system is stable, in this way a desirable behaviour of the trajectory should be as it is shown in Fig. 3.

![Nyquist Diagram](image)

Fig. 3. Desired stable Nyquist trajectory of the system with two open-loop unstable poles.

For the sake of simplicity on the frequency domain analysis, let us use an alternative representation of the PD controller given by,

\[ \bar{C}(s) = k_p(k_ds - 1), \]  
(12)
with \( k_p, k_d > 0 \). Notice that (12) is a non-minimum phase controller. The selection of this controller structure allows to us to find the best stability conditions in terms of the delay value.

Let us assume that the condition (8) holds, it could be selected a \( k_d \) gain for the PD controller such that,

\[ \tilde{k}_d = \frac{1}{a} + 1 - \frac{1}{c} \sum_{i=1}^{m} b_i - \frac{1}{d_i} - \bar{\theta} + \epsilon, \]  
(13)
with \( \epsilon \) being a sufficient small positive constant.

In order to satisfy the Nyquist stability criteria a frequency domain analysis of the system \( \bar{Q}(j\omega) = \bar{C}(j\omega)G_d(j\omega)G_r(j\omega) \) is required and performed in what follows. Thus, the phase equation of the system is,

\[ \Phi_{\bar{Q}}(\omega) = \arctan\left(\frac{\hat{\omega}}{a}\right) + \arctan\left(\frac{\hat{\omega}}{c}\right) - \sum_{i=1}^{n} \arctan\left(\frac{\hat{\omega}}{b_i}\right) \]  
(14)
- \sum_{i=1}^{n} \arctan\left(\frac{\hat{\omega}}{d_i}\right) - \arctan(\tilde{k}_d\hat{\omega}) - \omega\theta - \pi.

It can be seen that the phase begins in \( \Phi_{\bar{Q}}(0) = -\pi \) and its trajectory follows a negative direction, this could be demonstrated considering the derivative of the phase,

\[ \frac{d\Phi_{\bar{Q}}(\omega)}{d\omega} = \frac{a}{\omega^2 + a^2} + \frac{c}{\omega^2 + c^2} - \sum_{i=1}^{m} \frac{b_i}{\omega^2 + b_i^2} - \sum_{i=1}^{n} \frac{d_i}{\omega^2 + d_i^2} - \frac{\tilde{k}_d}{\omega^2 + 1} - \theta, \]  
(15)
for frequencies in the neighborhood of \( \omega \approx 0 \),

\[ \frac{d\Phi_{\bar{Q}}(\omega)}{d\omega} \bigg|_{\omega \approx 0} = \frac{1}{a} + \frac{1}{c} \sum_{i=1}^{m} \frac{1}{b_i} - \frac{1}{d_i} - \tilde{k}_d - \theta < 0. \]  
(16)

From (16), the starting angle of the system depends on the value of \( \epsilon \). It can be seen that for a \( \epsilon \rightarrow 0 \), the initial direction of the phase is slightly negative, which yields a crossover frequency \( \omega_c1 \rightarrow 0 \), where \( \omega_c1 \) is the first frequency \( \omega_c1 > 0 \) such that \( \Phi_{\bar{Q}}(\omega_c1) = -\pi \).

Moreover, there is a frequency \( \omega_s = \sqrt{ac} \) such that \( \Phi_{\bar{Q}}(\omega_s) > -\pi \) (see Lemma 1 which can be extended to the general case \( \Phi_{\bar{Q}}(\omega) \)).

Additionally, analyzing the phase derivative for high frequencies, namely \( \omega \approx \infty \), we can see that its trajectory is decreasing for high frequencies. In order to complete the analysis let us consider the magnitude equation given by,

\[ M_{\bar{Q}}(\omega) = K \sqrt{\frac{\tilde{k}_d^2}{(\omega_0^2 + 1)(\omega_0^2 + 1)} \prod_{i=1}^{m} \left(\frac{1}{\omega_0^2 + 1}\right) \prod_{i=1}^{n} \frac{1}{\omega_0^2 + 1}}, \]  
(17)
with \( K = \frac{\tilde{a}}{\tilde{c}} \prod_{i=1}^{m} \frac{1}{b_i} \prod_{i=1}^{n} \frac{1}{d_i} \), \( \tilde{a} = \frac{1}{\omega_0^2 + 1} \), \( \tilde{c} = \prod_{i=1}^{m} \left(\frac{1}{\omega_0^2 + 1}\right) \prod_{i=1}^{n} \frac{1}{\omega_0^2 + 1} \)

For small frequencies, the magnitude has a positive direction, which is easily concluded from:

\[ \frac{dM_{\bar{Q}}(\omega)}{d\omega} \bigg|_{\omega \approx 0} \approx \tilde{k}_d^2 - \frac{2}{a^2} - \frac{1}{c^2} - \sum_{i=1}^{m} \frac{1}{b_i^2} - \sum_{i=1}^{n} \frac{1}{d_i^2} > 0. \]  
(17)
Since the open loop system \( \bar{Q}(j\omega) \) is strictly proper, there is a decreasing magnitude for large frequencies. As it could
be noted, the selection of $\bar{k}_d$ results in a system with a positive gain margin, for a positive crossover frequency $\omega_c1 \to 0$, and a positive phase margin occurring at $\omega_m > 0$. Then, it can be concluded that the system $G_dG_r$ can be stabilized with a PD controller (12) choosing a derivative gain according to (13); and equivalently the recycling system (1) can be stabilized with a PD controller (6) in the control loop shown in Fig. 2.

**Remark 1.** An adequate choice of the derivative gain $k_d$ assures a Nyquist path with two encirclements in counter-clockwise direction, the system can be stabilized. In order to obtain a closed-loop stable system, the proportional gain $k_p$ should be selected such that:

$$\frac{ac\prod_{i=1}^{m}b_i\prod_{i=1}^{n}d_i}{\alpha F(\omega_{c1})} < k_p < \frac{ac\prod_{i=1}^{m}b_i\prod_{i=1}^{n}d_i}{\alpha F(\omega_{c1})},$$

where

$$F(\omega_{c1}) = \frac{k_d^{-2} \omega_{c1}^2 + 1}{(\frac{\omega_m^2}{\alpha^2} + 1)(\frac{\omega_c1^2}{\alpha^2} + 1)\prod_{i=1}^{m}(\frac{\omega_m^2}{\alpha^2} + 1)\prod_{i=1}^{n}(\frac{\omega_c1^2}{\alpha^2} + 1)}$$

and $\omega_{c1} > 0$ is denominated the first crossover frequency such that $\phi_Q(\omega_{c1}) = -\pi$.

The frequency domain analysis used in the proof of Theorem 1 can be applied to the case of the PID controller obtaining the same stability condition. In this way, the derived following result is stated.

**Corollary 1.** Consider the feedback control shown in Fig. 2. Then, the delayed recycling system given by (1) can be stabilized with a PID controller given by,

$$C(s) = K_p + \frac{K_i}{s} + K_Ds,$$

or

$$C(s) = k_p + \frac{k_d-s-1}{s} + 1,$$

(with $k_p, k_i, k_d > 0$) if and only if the condition (8) is satisfied.

**Remark 2.** An adequate selection of the derivative gain $k_d$ and the integral term $k_i$ in the PID controller given by (21) assures a Nyquist path with two counter-clockwise direction, i.e., the closed-loop stability warranty. In order to obtain a closed-loop stable system, the proportional gain $k_p$ should be selected such that,

$$\frac{ac\prod_{i=1}^{m}b_i\prod_{i=1}^{n}d_i}{\alpha \gamma(\omega_{c2})} < k_p < \frac{ac\prod_{i=1}^{m}b_i\prod_{i=1}^{n}d_i}{\alpha \gamma(\omega_{c1})},$$

with

$$\gamma(\omega_{cj}) = \sqrt{\frac{(-\omega_m^2 k_d + k_i)^2 + \omega_{cj}^2}{\omega_{cj}^2 + 1} \left(\frac{\omega_m^2}{\alpha^2} + 1\right) \left(\frac{\omega_cj^2}{\alpha^2} + 1\right)v}$$

for $j = 1, 2$.

where $v = \prod_{i=1}^{m}(\frac{\omega_m^2}{\alpha^2} + 1)\prod_{i=1}^{n}(\frac{\omega_cj^2}{\alpha^2} + 1)$ and $\omega_{c1}, \omega_{c2} > 0$ are denominated the first and second crossover frequencies respectively, such that $\Phi(\omega) = -\pi$, where

$$\Phi(\omega) = \arctan(\frac{\omega}{a}) + \arctan(\frac{\omega}{c}) - \sum_{i=1}^{m} \arctan(\frac{\omega}{b_i})$$

$$- \sum_{i=1}^{n} \arctan(\frac{\omega}{d_i}) - \arctan(\bar{k}_d \omega - \frac{k_i}{\omega}) - \omega \theta - \pi.$$

A simple procedure to determine the PD/PID controller parameters is proposed.

**Procedure 1.** (1) Identify the plant parameters $a$, $c$, $b_i$, $d_i$, $\alpha$, $\theta$.

(2) If the stability condition (8) holds, continue with step 3; otherwise, a PD or PID controller cannot stabilize the recycling delayed process (1) in the control scheme shown in Fig. 2.

(3) Compute the minimum stabilizing derivative gain $k_d$ for the PD/PID controller (given by (7)/(21)) according to (13).

(4) For the controller (7)/(21), search for a maximum $k_d$ such that the phase equation $\Phi_Q(\omega_{c1}) = -\pi$ (given by (14)), for some $\omega_{c1} > 0$ where $\omega_{m1} > \omega_{c1} > 0$ and $\omega_m$ is the frequency such that $M(\omega_m) = 1$.

(5) If a PD controller (7) is required, select a stabilizing $k_d$ from the obtained interval in Steps 3-4 and compute the stabilizing $k_p$ interval using (18).

(6) If a PID controller (21) is required, select a stabilizing $k_d$ from the obtained interval in Steps 3-4 and obtain the maximal $k_i (k_{i_{max}})$, starting from $k_i = 0$ such that $\Phi(\omega) > -\pi$ (given by (24)) for some $\omega > 0$. Then, select a $k_i$ into the interval $0 < k_i < k_{i_{max}}$, and with the selected $k_d$ and $k_i$, compute the stabilizing $k_p$ interval using (22).

4. NUMERICAL SIMULATION

**Example 1.** Consider a recycling system with

$$G_d(s) = \frac{2}{(s-0.5)(s+10)}e^{-0.1s},$$

$$G_r(s) = \frac{3}{(s-1)(s+9)}e^{-0.1s},$$

Here the design of the controller $C(s)$ is obtained. The proposed Procedure 1 is considered to design a PID controller. Therefore, the interval of the parameter controller $k_d$ is given by $2.4889 < k_d < 6.8$. For the simulation $k_d = 3.7$ is used, with this value of $k_d$ we obtain the interval of the $k_i$, as $0 < k_i < 0.155$. Using $k_d = 3.7$ and $k_i = 0.01$ the stabilizing $k_p$ interval is obtained, $5.7226 < k_p < 7.4626$. Therefore, the controller parameters $k_p = 6.5, k_i = 0.01$ and $k_d = 3.7$ are used to set the proposed controller given by (21). Fig. 4 shows the output signal behaviour for the nominal case (solid line) and when uncertainties into the both time-delays are considered, 50% in $\tau_1$ and +20% in $\tau_2$. Under these conditions, Fig. 5 exhibits the corresponding output control. From Figures, we can see that the closed-loop stability is preserved even when some uncertainties are regarded. Fig. 4 and Fig. 5 intend to
illustrate from a simple way the robustness characteristic associated to the proposed controller.

![Graph](image)

**Fig. 4. Output behaviour for Example 1**

**Fig. 5. Control signal in Example 1**

5. CONCLUSION

The analysis developed through this work gives as result a stabilization strategy for a special kind of recycling system. The proposed stabilization scheme allows the use two different controllers, i.e., a Proportional-Derivative controller and a Proportional-Integral-Derivative controller. Explicit stability conditions are obtained in terms of the system model parameters. Even when a classical PID controller has the characteristic of step tracking reference and step disturbance rejection, in this work the used PID controller only assures closed-loop stability and perhaps this is a negative consequence of the proposed stabilizing strategy. However, future extension of this work considers the step reference tracking as well as the step disturbance rejection. The importance of this work arises from the fact that the proposed strategy simplify the stability problem with two different time-delay terms to a problem with one time-delay. The present research should be complemented by the design of the estimation stage, which will be regarded as a future work.

**REFERENCES**


**Appendix A. AUXILIAR LEMMA**

**Lemma 1.** Considering the phase equation,

\[ \Phi_A(\omega) = \arctan\left(\frac{\omega}{a}\right) + \arctan\left(\frac{\omega}{c}\right) - \arctan\left(\tilde{k}_d\omega\right) - \omega\theta - \pi \]  

Then, there exists a frequency \( \omega_k = \sqrt{ac} \) such that \( \Phi_A(\omega_k) > -\pi \) if \( \tilde{k}_d \) is given according to,

\[ \tilde{k}_d = \frac{1}{a} + \frac{1}{c} - \theta + \epsilon, \]  

**Proof.** Let us consider the phase equation (A.1), with \( \tilde{k}_d \) according to (A.2). Taking into account the phase contribution of the two unstable time constants, \( \arctan\left(\frac{\omega}{a}\right) + \arctan\left(\frac{\omega}{c}\right) \) A trigonometric identity for the sum of the two \( \arctan() \) functions is given by,

\[ \arctan\left(\frac{\omega}{a}\right) + \arctan\left(\frac{\omega}{c}\right) = \arctan\left(\frac{\omega(a+c)}{ac-\omega^2}\right). \]
Then, the phase in $\omega_s$ is:
\[
\arctan\left(\frac{\omega_s}{a}\right) + \arctan\left(\frac{\omega_s}{c}\right) = \frac{\pi}{2}.
\]

Considering the negative terms in the phase equation (A.1) as $\Phi(\omega) = \arctan(\bar{k}_d\omega) + \theta\omega$ and considering the frequency $\omega_s$, an extensive calculation routine allows to conclude,
\[
\Phi(\omega_s)|_{\theta \rightarrow \frac{1}{a} + \frac{1}{c} - \sqrt{\frac{1}{a^2} + \frac{1}{c^2}}} < \frac{\pi}{2},
\]
for $a, c$ being real positive numbers. Moreover, the derivative of $\Phi(\omega_s)$ with respect to the time-delay is,
\[
\left.\frac{d\Phi(\omega)}{d\theta}\right|_{\omega=\omega_s} = \frac{1 - \frac{1}{ac(\frac{1}{a} + \frac{1}{c} - \theta)^2 + 1}}{\sqrt{\frac{1}{ac}}},
\]
which is positive for all $\theta > 0$. Then, the function $\Phi(\omega_s)$ is monotonically increasing with respect to the delay size. Therefore, we can assure that,
\[
\Phi(\omega_s) = \arctan\left(\frac{\omega_s}{a}\right) + \arctan\left(\frac{\omega_s}{c}\right) - \arctan(\bar{k}_d\omega_s) - \omega_s\theta - \pi > -\pi
\]
for a delay such that $0 < \theta < \frac{1}{a} + \frac{1}{c} - \sqrt{\frac{1}{a^2} + \frac{1}{c^2}}$. ■